Part 2

Area & Definite Integrals

2.1 Segment: Area Bound by a Curve; Definite Integral Power Rule

2.1.1 Area Bound by a Curve & Accumulated Values

Many situations involve accumulating values, i.e., a summation. For example, profits per month for Joanie Cakes are recorded in the following table:

<table>
<thead>
<tr>
<th>Month $t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit $p$ (ten thousands of dollars)</td>
<td>25</td>
<td>36</td>
<td>15</td>
<td>22</td>
<td>18</td>
<td>41</td>
</tr>
</tbody>
</table>

How much total profit did Joanie Cakes for the six months given in the table?

The total profit is the sum (accumulation) of the profit per month $p(t)$:
25 + 36 + 15 + 22 + 18 + 41 = 157, $1,570,000.

We can graph the monthly profit.

The area of each rectangle in the graph gives the profit for a month. The total area (sum) is the total profit for six months.

In the previous example, the values of $p$ are fixed for large units of time (a discrete function). What happens if the function values could change at every instance (a continuous function)? The area under the curve still represents the accumulation or sum, but the computation cannot be done with a simple sum of rectangle areas.

A container is to be filled with water from a hose. At the beginning, the tap controlling the water flow is turned off and the container contains 2 liters of water. The tap is slowly turned on and then slowly turned off, so that the rate of flow of water changes at every instant. The variable rate of flow in liters per minute is given by $r = -t^2 + 6t$ where $t$ is time in minutes. How much water is in the container after six
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minutes?

We need to find the accumulation of water at \( r \) liters per minute for six minutes. If we graph \( r = -t^2 + 6t \) as a function of \( t \), the area bounded under the curve between \( t = 0 \) and \( t = 6 \) would represent the accumulation of water.

How can we add up (find the sum) of \( r(t) = -t^2 + 6t \) for \( 0 \leq t \leq 6 \) (the area under the curve). In the previous example, we could find the sum because we knew the formula for areas of rectangles, now we have an unusual shape without a standard formula from geometry. We first present some notation and formulas and second compute the unusual shape area.

The general area under a curve over an interval \( a \leq t \leq b \) is called a \textit{definite integral}. The integral notation uses the old german \( s \) for sum.

\[
\int_{a}^{b} r(t) \, dt \quad \text{represents the definite integral (area) for } r(t) \text{ over the interval } a \leq t \leq b.
\]
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2.1.2 Basic definite integral formulas

Sum Rule: \( \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \),

Constant Coefficient Rule: \( \int_a^b cf(x) \, dx = c\int_a^b f(x) \, dx \),

Power Rule: For \( n \neq -1 \), \( \int_a^b x^n \, dx = \left[ \frac{1}{n+1} x^{n+1} \right]^b_a = \frac{1}{n+1} (b)^{n+1} - \frac{1}{n+1} (a)^{n+1} \).

Example 1

\[
\int_1^5 x^3 \, dx = \left[ \frac{1}{3+1} x^{3+1} \right]^5_1 = \frac{1}{4} (5)^4 - \frac{1}{4} (1)^4 = 156.
\]

Example 2

\[
\int_1^5 x^{-3} \, dx = \left[ \frac{1}{-3+1} x^{-3+1} \right]^5_1 = \frac{1}{-2} (5)^{-2} - \frac{1}{-2} (1)^{-2} = 2.96.
\]

Example 3

\[
\int_1^5 x^3 + x^{-3} \, dx = \left[ \frac{1}{3+1} x^{3+1} + \frac{1}{-3+1} x^{-3+1} \right]^5_1 = 1.56 + 2.96 = 4.52
\]

Example 4

\[
\int_1^5 7x^3 \, dx = 7 \left[ \frac{1}{3+1} x^{3+1} \right]^5_1 = 7(1.56) = 10.92
\]

Example 5

\[
\int_1^5 7x^3 - 2x^{-3} \, dx = \left[ 7 \left( \frac{1}{3+1} x^{3+1} \right) - 2 \left( \frac{1}{-3+1} x^{-3+1} \right) \right]^5_1 = 7 (1.56) - 2 (2.96) \approx 11.13
\]
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In the examples so far, the values of the function being integrated is positive. When the values of the function are negative, the integral accumulates the area, but with a minus sign (negative area).

Example 6

\[ \int_{1}^{5} -3x^3 \, dx = -3 \cdot (1.56) = -4.68. \] The (positive) area is given by
\[ \int_{1}^{5} |-3x^3| \, dx = 4.68. \]

\[ y = -3x^3 \quad y = |-3x^3| \]

Lets return to the example with water accumulating at a rate given by \( r = -t^2 + 6t \). The total amount of water accumulated would be
\[ 2 + \int_{0}^{6} (-t^2 + 6t) \, dt = 2 + \left[ -\frac{1}{2+1} t^{2+1} + 6 \left( \frac{1}{1+1} t^{1+1} \right) \right]_{0}^{6} = 2 + \left[ -\frac{1}{3} t^3 + 3t^2 \right]_{0}^{6} = 2 + \left( \left[-\frac{1}{3} (6)^3 + 3 (6)^2 \right] - \left(-\frac{1}{3} (0)^3 + 3 (0)^2 \right) \right) = 36 \text{ liters.} \]