Part 3

Limits

3.1 Segment: Factoring, Zeros and Roots, Fractions

3.1.1 Factoring Quadratic Polynomials

Example 1

Factor $x^2 + 5x + 6$

The coefficient of $x^2$ is one, so the factorization has the form $(x + a) (x + b)$

where $a$ and $b$ are factors of 6. In addition $a + b = 5$.

The factorizations of 6 are $(1)(6)$ and $(2)(3)$; $1 + 6 = 7$, $2 + 3 = 5$.

We choose $a = 2$ and $b = 3$.

$x^2 + 5x + 6 = (x + 2)(x + 3)$
This example uses a monic polynomial, i.e., the coefficient of $x^2$ is one. In this case factoring is accomplished by finding factors of the constant term that add to the coefficient of $x$.

**Example 2**

Factor $15x^2 + 28x + 12$

The coefficient of $x^2$ is not one, so the factorization has the form $(ax + b)(cx + d)$ where, $a$ and $c$ are factors of the coefficient of $x^2$ that is 15, and, $b$ and $d$ are factors of the constant term 12.

The complete factorization of 15 is $3 \cdot 5$

The complete factorization of 12 is $2 \cdot 2 \cdot 3$

Line up the factors of 15 and 12.

$3 \cdot 5 \cdot 2 \cdot 2 \cdot 3$ cut the list of factors into three parts; in the first part only factors of 15 and the third part only factors of 12.

Compute the product of the first and third part, then add the product of the middle part.

Some possible divisions:

$[3] \cdot [5 \cdot 2 \cdot 2] \cdot [3]$  \quad  (3) (3) + (5 \cdot 2 \cdot 2) = 9 + 20 = 29

$[3] \cdot [5 \cdot 2] \cdot [2 \cdot 3]$  \quad  (3) (2 \cdot 3) + (5 \cdot 2) = 18 + 10 = 28

Notice that 28 is the coefficient of $x$ in $15x^2 + 28x + 12$. The correct
partitioning of the factors is \([1 \cdot 3] \cdot [5 \cdot 2] \cdot [2 \cdot 3 \cdot 1]\).

In the factorization \((ax + b)(cx + d)\), \(a = (1 \cdot 3) = 3\), \(bc = (5 \cdot 2)\),
and \(d = (2 \cdot 3 \cdot 1) = 6\).

In the product \(bc = (5 \cdot 2)\), \(c\) is the set of factors from 15, in this case 5, and \(b\) is the set factors from 12, in this case 2.

\[15x^2 + 28x + 12 = (3x + 2)(5x + 6)\]

**Example 3**

Factor \(15x^2 - 28x + 12\)

Notice that \(15x^2 - 28x + 12\) only differs from the last example in that 28 has been changed to 28.

We still use the same set of factors from 15 and 12, and the same partition of the factors:

\[3 \cdot [5 \cdot 2] \cdot [2 \cdot 3]\]

\((3)(2 \cdot 3) + (5 \cdot 2) = 18 + 10 = 28\).

Now the factorization is \((3x - 2)(5x - 6)\), where \((-2)(-6)\) replaces \((2)(6)\) since \(-28\) replaced 28.

**Example 4**

Factor \(15x^2 + 8x - 12\)

In this problem, as in the last two examples, we use the factors of
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15 and 12:

\[ [3] \cdot [5 \cdot 2] \cdot [2 \cdot 3] \quad (3)(2 \cdot 3) - (5 \cdot 2) = 18 - 10 = 8 \]

The sum in the previous example is replaced by a difference when 15 and −12 are opposite sign.

The factorization is \((3x - 2)(5x + 6)\).

Example 5

Factor \(4x^2 - 25\)

This is an example of difference of squares factorization: \((2x - 5)(2x + 5)\)

Example 6

It is not always necessary to use whole number coefficients. Factor

\[ x^2 - 7 \]

\[ x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7}) \]

3.1.2 Zeros and Roots

A root of an expression, such as \(x^2 - 5x + 6\), is a value of the variable that makes the expression equal to zero. The value 3 is a root of \(x^2 - 5x + 6\) since \((3)^2 - 5(3) + 6 = 0\).
We may solve for roots by setting the expression equal to zero, \( x^2 - 5x + 6 = 0 \),
\((x - 2)(x - 3) = 0, x - 2 = 0 \text{ or } x - 3 = 0, x = 2, 3 \) are the roots.

For a function, such as, \( y = x^2 - 5x + 6 \), the roots of the expression \( x^2 - 5x + 6 \)
are referred to as zeros of the function, i.e., the values of \( x \) that give \( y = 0 \). The zeros
of a function are also the \( x \)-intercepts of the function.

If \( r_1 \) and \( r_2 \) are the roots of a quadratic then \( x - r_1 \) and \( x - r_2 \) are factors.
We can always find the roots of a quadratic using the quadratic formula: 

\[ ax^2 + bx + c = 0, \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

So \( ax^2 + bx + c = a \left( x - \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left( x - \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right). \]

**Example 7**

Factor \( x^2 - 3x - 7 \)

\[ x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2(1)} = \frac{3 \pm \sqrt{37}}{2}, \]

\[ x^2 - 3x - 7 = \left( x - \frac{3 + \sqrt{37}}{2} \right) \left( x - \frac{3 - \sqrt{37}}{2} \right). \]

### 3.1.3 Simplifying Algebraic Fractions, Rationalizing

**Example 8**

Reduce \( \frac{3x - 9}{3} \)

\[ \frac{3x - 9}{3} = \frac{3(x - 9)}{3} = x - 9 \]

**Example 9**

Reduce \( \frac{2x - 6}{3} \)

\[ \frac{2x - 6}{3} = \frac{2(x - 3)}{3} \text{ does not reduce.} \]

**Example 10**
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Reduce $\frac{x^2-5x+6}{x-3}$

$\frac{x^2-5x+6}{x-3} = \frac{(x-3)(x-2)}{x-3} = x - 2$

Example 11

Reduce $\frac{x^2-5x+6}{x+3}$

$\frac{x^2-5x+6}{x+3} = \frac{(x-3)(x-2)}{x+3}$ does not reduce

Example 12

Simplify $\frac{1}{x-9} + \frac{x^2}{x^2-5x+6}$

$\frac{1}{x^2-9} + \frac{x+2}{x^2-5x+6} = \frac{1}{(x-3)(x+3)} + \frac{x+2}{(x-3)(x-2)} = \frac{1}{(x-3)(x+3)} \cdot \frac{x-2}{x-2} + \frac{x+2}{(x-3)(x-2)}$

$\frac{x+3}{x+3} = \frac{x-2}{(x-3)(x-2)(x+3)} + \frac{(x+2)(x+3)}{(x-3)(x-2)(x+3)} = \frac{x-2+(x+2)(x+3)}{(x-3)(x-2)(x+3)} = \frac{x^2+6x+4}{(x-3)(x-2)(x+3)}$

3.2 Segment: Concepts of infinity & zero; Limits from graphs

3.2.1 Limits and Graphs

Sandra is climbing a hill. She cannot see past a point on the hill.