Chapter 3

Limits

3.1 Prerequisites: Factoring, Reducing Algebraic Fractions; Roots and Zeros

A product is made up of factors that are multiplied. For example, the product $2xy$ has three factors: $2$, $x$, and $y$. A sum is made up of terms that are added. For example, the sum $2 + 3x^2$ has two terms: $2$ and $3x^2$. An algebraic product is sometimes equivalent to an algebraic sum. Using the distributive law, the product $2x(2x + 7y)$ is equivalent to $(2x)(2x) + (2x)(7y)$, or $4x^2 + 14xy$. The expression $2x(2x + 7y)$ represents a product of factors and the expression $4x^2 + 14xy$ represents a sum of terms. To properly manipulate these two expressions it is of upmost importance to distinguish between the two representations. In the representation $2x(2x + 7y)$, $2x$ is a factor of the entire expression. In the representation $4x^2 + 14xy$, $y$ is a factor of just one term, not a factor of the entire expression.

3.1.1 Factoring expressions written as a sum of terms

The expression $6x^3 + 3x^2 + 9x$ is written as a sum of terms. Can $6x^3 + 3x^2 + 9x$ be written as a product of factors, in other words, can the expression be factored. In the process of factoring a sum of terms, first consider a common factor. A common factor is a factor of each of the individual terms in the sum. Notice that $3x$ is a factor of each of the terms $6x^3$, $3x^2$, and $9x$, thus, $3x$ is a common factor for the expression $6x^3 + 3x^2 + 9x$. The expression can be written $(3x)(2x^2) + (3x)(x) + (3x)(3)$. The distributive law allows the common factor to be factored out, $(3x)(2x^2 + x + 3)$.

Always check for a common factor as the first step in factoring.

The product of two binomial factors, such as $(3x - 4)(2x + 3)$, can be mul-
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tplied out (expanded) to a sum of four terms, the products of each of the each of the terms of the first factor with each of the terms of the second factor. That is, \((3x - 4)(2x + 3) = (3x)(2x) + (3x)(3) + (-4)(2x) + (-4)(3) = 6x^2 + x - 12\)
a trinomial. A product has been expanded to a sum that is equivalent. Now we wish to start with a trinomial sum and factor to a product of two binomials.

Let’s start with \(6x^2 + x - 12\) and reconstruct the factors \((3x - 4)(2x + 3)\). In the factor \((3x - 4)\), the coefficient of \(x\), 3, is a factor of the coefficient of \(x^2\) in the sum \(6x^2 + x - 12\) and the constant, -4, is a factor of the constant in \(6x^2 + x - 12\). If we did not already known that the factors of \(6x^2 + x - 12\) are \((3x - 4)\) and \((2x + 3)\), we would at least know factors are of the form \(\mu x + \nu\) where \(\mu\) is a factor of 6 and \(\nu\) is a factor of 12. Proceed by check all products, such as \((2x + 6)(3x - 2) = 6x^2 + 14x - 12\), where coefficients of \(x\) multiply to 6 and the constants multiply to -12. Continue check all such products until the product \((3x - 4)(2x + 3) = 6x^2 + x - 12\) is discovered. In this process, a factorization is discovered or all possibilities are exhausted and there is no factorization.

Example: Factor \(15x^2 + 28x + 12\).

Factors of 15: 1, 3, 5, 15. Factors of 12: 1, 2, 3, 4, 6, 12. Begin trying binomial factors such as \((5x + 3)\), using a positive factor of 15 for the coefficient of \(x\) and positive or negative factor of 12 for the constant. Check the product: \((5x + 3)(3x + 5) = 15x^2 + 34x + 15\), if the result is the correct trinomial stop, if not, continue until a correct product is found.

\[
\begin{align*}
(x - 4)(15x - 3) &= 15x^2 - 63x + 12, \text{ not correct,} \\
(3x + 4)(5x + 3) &= 15x^2 + 29x + 12, \text{ not correct,} \\
(15x + 1)(x + 12) &= 15x^2 + 181x + 12, \text{ not correct,} \\
(5x + 4)(3x + 3) &= 15x^2 + 27x + 12, \text{ not correct,} \\
(3x + 2)(5x + 6) &= 15x^2 + 28x + 12, \text{ correct, the proper product is found; the factorization is complete.}
\end{align*}
\]

Example: Factor \(x^2 - 5x - 6\).

Factors of 1: 1. Factors of 6: 1, 2, 3, 6. Since there is only one choice for the factors of 1, find the sum or difference of factors 6, multiplying to 6, that equals the coefficient of \(x\), -5: \(2 + 3 = 5\), \(2 - 3 = -1\), \(1 - 6 = -5\) stop. Factorization: \((x + 1)(x - 6) = x^2 - 5x - 6\).

3.1.2 Reducing Algebraic Fractions

An algebraic fraction, such as \(\frac{12x^3}{4x}\), can sometimes be reduced to a simpler form.

For \(\frac{12x^3}{4x}\), we find common factors to the numerator and denominator \(\frac{(3x)(4x^2)}{(3x)(3)}\).

Then \(\frac{(3x)(4x^2)}{(3x)(3)} = \frac{4x^2}{3}\). In this example, both the denominator and numerator consist of only one term.

For an algebraic fraction where the numerator or denominator consists of more than one term, we proceed in the same way, find common factors for the numerator and denominator, then reduce the fraction. We must be very careful to make sure that a common factor is a factor of an entire numerator and an entire denominator not just one term of a numerator or denominator.
Example: Reduce \( \frac{3x-9}{3} \).
\[
\frac{3x-9}{3} = \frac{3(x-3)}{3} = \frac{x-3}{1} = x-3.
\]
Example: Reduce \( \frac{x^2-5x+6}{x-3} \).
\[
\frac{x^2-5x+6}{x-3} = \frac{(x-3)(x-2)}{x-3} = \frac{x-2}{1} = x-2.
\]
Example: Reduce \( \frac{x^2+4x+4}{4x^2-16} \).
\[
\frac{x^2+4x+4}{4x^2-16} = \frac{(x+2)(x+2)}{4(x-2)(x+2)} = \frac{x+2}{4(x-2)} \cdot \frac{x+2}{4(x-2)} = \frac{x+2}{4(x-2)}.
\]