3.3.3 Basic Rules for Limits

The following rules hold for limits when the quantities exist:

\[
\lim_{x \to \beta} f(x) + \lim_{x \to \beta} g(x) = \lim_{x \to \beta} (f(x) + g(x))
\]

\[
\lim_{x \to \beta} f(x) \cdot \lim_{x \to \beta} g(x) = \lim_{x \to \beta} (f(x) \cdot g(x))
\]

\[
\frac{\lim_{x \to \beta} f(x)}{\lim_{x \to \beta} g(x)} = \lim_{x \to \beta} \left( \frac{f(x)}{g(x)} \right).
\]

Example 21

Let \( \lim_{x \to b} f(x) = 3 \) and \( \lim_{x \to b} g(x) = -4 \).

\[
\lim_{x \to b} (f(x) + 3g(x)) = 3 + 3(-4) = -9
\]

\[
\lim_{x \to b} \left( \frac{7f(x)}{5g(x)} \right) = \frac{7(3)}{5(-4)} = -\frac{21}{20}
\]

3.4 Segment: Limits at Infinity; Continuity

3.4.1 Limits at Infinity

We have seen functions \( f(x) \) that tend to become large without bound near a value of \( x \) and assigned the concept of infinity to this situation, \( \lim_{x \to 3} f(x) = \infty \).

Now let \( x \) become large without bound, \( x \to \infty \), and look at the tendency of \( f(x) \): \( \lim_{x \to \infty} f(x) \).

Example 22
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\( y = f(x) = 2x^3 \) grows without bound when \( x \) grows without bound:

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2x^3 = \infty.
\]

Example 23

\( y = f(x) = \frac{3}{x} \) gets smaller and smaller (close to zero) as \( x \) gets larger and larger:

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3}{x} = 0.
\]

Example 24

\[
\lim_{x \to \infty} 2 - \frac{3}{x} = 2 - 0 = 2.
\]

For the polynomial \( x^3 - 2x + 5 \), when we substitute a very large value of \( x \), such as, \( 10^6 \), the value is almost the same as just the leading term \( x^3 \): \((10^6)^3 - 2(10^6) + 5 = 9999999998000005\), \((10^6)^3 = 10000000000000000\), these quantities differing only after 11 places. The point is, for very large values of \( x \), a polynomial is very nearly the same as its leading term. So for limits, such as, \( \lim_{x \to \infty} \frac{3x^2 - 2x + 5}{6x^2 + 3x - 7} \), we may replace the numerator and denominator by their leading terms, \( \lim_{x \to \infty} \frac{3x^2 - 2x + 5}{6x^2 + 3x - 7} = \lim_{x \to \infty} \frac{3x^2}{6x^2} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2} \). It is very, very important to only use this process when \( x \to \infty \) or \( x \to -\infty \).

Example 25

\[
\lim_{x \to \infty} \frac{2x + 5}{3 - 7x} = \lim_{x \to \infty} \frac{2x}{-7x} = \lim_{x \to \infty} \frac{2}{-7} = -\frac{2}{7}.
\]
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Example 26

\[
\lim_{x \to \infty} \frac{2x^2 + 2x - 1}{5x + 3} = \lim_{x \to \infty} \frac{2x^2}{5x} = \lim_{x \to \infty} \frac{2x}{5} = \infty.
\]

Example 27

\[
\lim_{x \to \infty} \frac{2x^2 - 6x + 5}{6x^4 - 7x - 3} = \lim_{x \to \infty} \frac{2x^2}{6x^4} = \lim_{x \to \infty} \frac{1}{3x^2} = 0.
\]

3.4.2 Continuity

When we look at the graph of a function like \( y = 2x^2 - 4 \), there are no holes or jumps. We can move continuously across any point on the graph.
A function $y = f(x)$ is *continuous* at $x = b$ if $f(b) = \lim_{x \to b} f(x)$, the function is defined at $b$ and the approach to $b$ matches the value at $b$.

Polynomials, rational functions, roots, exponentials, and logarithms are continuous at each point where they are defined, we calculate the limit merely by evaluating the function.

**Example 28**

The function $f(x) = \frac{2x-3}{x+1}$ is continuous at each point where it is
defined: $x \neq -1$. $f(-1)$ is not defined so $f$ cannot be continuous, furthermore $\lim_{x \to -1} f(x)$ does not exist. The function $f$ is continuous on the interval $(-\infty, -1)$ and on the interval $(-1, \infty)$.

Example 29

The function $f(x) = \frac{2x + 2}{x + 1}$ is continuous at each point where it is defined: $x \neq -1$. $f(-1)$ is not defined so $f$ cannot be continuous, but $\lim_{x \to -1} f(x) = 2$ does exist. The function $f$ is continuous on
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the interval \((-\infty, -1)\) and on the interval \((-1, \infty)\). The graph of \(f\) has a hole at \(x = -1\).

Example 30

The function \(f(x) = \frac{|2x+2|}{x+1}\) is continuous at each point where it is defined: \(x \neq -1\). \(f(-1)\) is not defined so \(f\) cannot be continuous, but \(\lim_{x \to -1^-} f(x) = -2\) \(\lim_{x \to -1^+} f(x) = 2\) The function \(f\) is continuous on the interval \((-\infty, -1)\) and on the interval \((-1, \infty)\). The graph
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of \( f \) has a jump at \( x = -1 \).

For each \( x \), the greatest integer function \( g(x) = \lfloor x \rfloor \) is defined to be the greatest integer less than \( x \).

\[
\lfloor 2.3 \rfloor = 2, \lfloor 0.3 \rfloor = 0, \lfloor \pi \rfloor = 3, \lfloor -2.3 \rfloor = -3, \lfloor -\pi \rfloor = -4, \lfloor 2 \rfloor = 2, \lfloor -2 \rfloor = -2, \text{ and } \lfloor 0 \rfloor = 0.
\]
\[
\lim_{x \to -1^-} [x] = -2, \quad \lim_{x \to -1^+} [x] = -1, \quad \lim_{x \to 3^-} [x] = 2, \quad \lim_{x \to -1^+} [x] = 3
\]

The function \( g(x) = [x] \) is not continuous at each integer, but is continuous between each integer; it is continuous on the intervals \( (n, n + 1) \) for each integer \( n \).