Part 7

Related Rates; Optimization

7.1 Segment: Related Rates: Part 1

Rate is a change in a quantity with respect to a change in time, i.e., the derivative with respect to time. Speed is the rate of change in distance with respect to time. Interest rate is a rate of change of money with respect to time. The rate at which oxygen is used up in a sealed room is the change in volume with respect to time.

Many quantities are related so that the rate of change of one affects the second. Changes in the amount of air blown into a balloon (volume) affects a change in the surface (area) of the balloon. This gives rise to related rates: the rate of change in volume and the rate of change in the surface area.

Example 1
PART 7

The radius of a circular oil slick is increasing at 50 feet per hour. When the oil slick is 1000 feet across, how fast is the area of the oil slick increasing?

Let $r$ be the radius and $A$ be the area of the oil slick. Both $r$ and $A$ are change with respect to time and they are related by $A = \pi r^2$.

The relation $A = \pi r^2$ also gives a relation for rates: $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

When $r = 1000$, $\frac{dA}{dt} = 2\pi (1000) (50) = 100000\pi \frac{ft^2}{hour}$.

Example 2

A cylindrical bucket is being filled with water from a hose running at 1/3 liter per minute. The diameter of the bucket is 40 centimeters. How fast is the height of the water rising in the bucket?

Let $h$ be the height of the water and $V$ be the volume of the water in the bucket. The fill rate is the rate of change of volume $\frac{dV}{dt}$. Volume and height are related by the formula $V = 400\pi h$.

$$\frac{d}{dt}(V) = \frac{d}{dt}(400\pi h), \quad \frac{dV}{dt} = 400\pi \frac{dh}{dt}, \quad \left( \frac{\frac{1000}{1 \text{ min}}}{\text{L}} \right) \left( \frac{1000 \text{ cm}^3}{1 \text{ L}} \right) = 400\pi \frac{dh}{dt},$$

$$\frac{dh}{dt} = 1.2\pi \frac{cm}{\text{min}}.$$

Example 3
A spherical weather balloon is being filled at 1.5 m$^3$/min. The balloon will be released when it reaches a diameter of 10 meters. How fast will the diameter be increasing when balloon is ready to release?

Let $V$ be the volume of the balloon and $r$ be the radius of the balloon then $V = \frac{4}{3}\pi r^3$.

Now \( \frac{d}{dt} (V) = \frac{d}{dt} \left(\frac{4}{3}\pi r^3\right) \), \( \frac{dV}{dt} = \frac{4}{3}\pi (3r^2 \frac{dr}{dt}) = 4\pi r^2 \frac{dr}{dt} \), 1.5 = 4\pi r^2 \frac{dr}{dt}.

At \( r = 5 \), 1.5 = 4\pi (5)^2 \frac{dr}{dt} \cdot \frac{dr}{dt} = \frac{1.5}{100\pi} = \frac{0.015}{\pi} \text{ m/min.}

\[ \text{7.2 Segment: Related Rates: Part 2} \]

\[ \text{Example 4} \]

The sales $s$ in hundred thousands of dollars at a company depend on the amount of advertising dollars $x$ in thousands of dollars according to the model $s = 10 - 10e^{-0.05x^2}$. Dollars are being invested in advertising at 2 thousand dollars per day. How fast are sales rising when five thousand dollars are invested in advertising?

\[ \frac{ds}{dt} = \frac{d}{dt} \left(10 - 10e^{-0.05x^2}\right), \frac{ds}{dt} = -10(-0.05)(2x)e^{-0.05x^2} \frac{dx}{dt} = xe^{-0.05x^2} \frac{dx}{dt} \]

When $x = 5$ and $\frac{dx}{dt} = 2$, $\frac{ds}{dt} = 5e^{-0.05(2)^2} (2) = 10e^{-0.2} \approx 8.19 \text{ hundred thousand dollars per day.}$