Part 8

Indefinite Integrals

8.1 Segment: Indefinite Integrals

We have seen that the definite integral $\int_{a}^{b} f(x)dx$ is the sum of $f(x)$ from $x = a$ to $x = b$. Now we are going to study the indefinite integral—the definite and indefinite integrals are come from two separate concepts. But, surprisingly, the two are very related.

An indefinite integral, also called the antiderivative, of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Example 1

$$f(x) = x^3$$

We know that the derivative of $g(x) = x^4$ is $g'(x) = 4x^3$. The
derivative of derivative of \( g \) only varies from \( f(x) \) by the constant factor 4. If we divide by 4 we should obtain \( f(x) = x^3 \). For \( F(x) = \frac{1}{4}x^4 \), \( F'(x) = x^3 = f(x) \); \( F(x) \) is an indefinite integral of \( f(x) \).

The function \( G(x) = \frac{1}{4}x^4 + 25 \) is also an indefinite integral of \( f(x) \);
\[
G'(x) = x^3 = f(x).
\]

If \( F(x) \) is an indefinite integral of \( f(x) \) then, for any constant \( C \), \( F(x) + C \) is also an indefinite integral of \( f(x) \). Notationally, we write \( \int f(x)\,dx = F(x) + C \), indicating that there a many possible integral functions.

For example, \( \int x^3\,dx = \frac{1}{4}x^4 + C \).

Since indefinite integrals are the inverse of derivatives, every derivative formula is also an integral formula.

The derivative of a power \( f(x) = x^n \), \( f'(x) = nx^{n-1} \), gives rise to the integral of a power \( f(x) = x^n \), \( \int f(x)\,dx = \frac{1}{n+1}x^{n+1} + C \) for \( n \neq -1 \).

The derivative of \( f(x) = \ln x \), \( f'(x) = x^{-1} = \frac{1}{x} \), gives rise to the integral of \( f(x) = x^{-1} \), \( \int f(x)\,dx = \ln x + C \).

The derivative of \( f(x) = e^x \), \( f'(x) = e^x \), gives rise to the integral of \( f(x) = e^x \), \( \int f(x)\,dx = e^x + C \).

The derivative of \( f(x) = 7^x \), \( f'(x) = 7^x \ln 7 \), gives rise to the integral of \( f(x) = 7^x \), \( \int f(x)\,dx = \frac{7^x}{\ln 7} + C \).
PART 8

The Fundamental Theorem of Calculus tells us that if $F(x)$ is an indefinite integral of $f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$. We may also use the notation $F(x)|_a^b = F(b) - F(a)$.

Example 2

$$\int_2^5 x^3\,dx = \frac{1}{4}x^4\,|_2^5 = \left(\frac{1}{4}\,5^4\right) - \left(\frac{1}{4}\,2^4\right) = \frac{625}{4}.$$

Example 3

Find the function $f(x)$ such that $f'(x) = 3x^2 - 5$ and $f(2) = 12$.

$$f(x) = \int f'(x)\,dx = \int (3x^2 - 5)\,dx = x^3 - 5x + C. \quad 12 = f(2) = 2^3 - 5(2) + C, \quad C = 14.$$ 

$$f(x) = x^3 - 5x + 14.$$

Example 4

Find the function $f(x)$ such that $f''(x) = 3x^2 - 5$, $f'(2) = 0$ and $f(2) = 4$.

$$f'(x) = \int f''(x)\,dx = \int (3x^2 - 5)\,dx = x^3 - 5x + C, \quad 0 = f'(2) = 0^3 - 5(0) + C, \quad C = 0.$$ 

$$f'(x) = x^3 - 5x.$$ 

$$f(x) = \int f'(x)\,dx = \int (x^3 - 5x)\,dx = \frac{1}{4}x^4 - \frac{5x^2}{2} + C, \quad 4 = f(2) =$$

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\[
\frac{1}{4} (2)^4 - \frac{5(2)^2}{2} + C, \quad 4 = 4 - 10 + C, \quad C = 10.
\]

\[
f(x) = \frac{1}{4}x^4 - \frac{5x^2}{2} + 10.
\]

### 8.2 Segment: Integration by Substitution

The chain rule for derivatives states that the derivative of \( f(u(x)) \) with respect to \( x \) is given by \( f'(u) \cdot u'(x); \frac{df(u(x))}{dx} = f'(u) \cdot u'(x) \). We can turn this into the substitution rule for integrals: \( f(u(x)) = \int \frac{df(u(x))}{dx} dx = \int f'(u) \cdot u'(x) dx \).

**Example 5**

\[
\int \sqrt{4x^2 - x} (8x - 1) \, dx
\]

Let \( u(x) = 4x^2 - x \) then \( u'(x) = 8x - 1 \) and \( \int \sqrt{4x^2 - x} (8x - 1) \, dx = \int \sqrt{u} \cdot u'(x) \, dx \). Now \( \int \sqrt{u} \, du = \int u^{1/2} du = \frac{u^{3/2}}{3/2} + C = \frac{2u^{3/2}}{3} + C \), so

\[
\int \sqrt{4x^2 - x} (8x - 1) \, dx = \frac{2(4x^2 - x)^{3/2}}{3} + C.
\]

**Example 6**

\[
\int x^2 e^{x^3} \, dx
\]

Let \( u(x) = x^3 \) then \( u'(x) = 3x^2 \) and \( \int x^2 e^{x^3} \, dx = \frac{1}{3} \int 3x^2 e^{x^3} \, dx = \frac{1}{3} \int e^u u'(x) \, dx = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C \).