Ordered Fields: Axioms and Basic Properties

Dr. Lance Nielsen
Creighton University Department of Mathematics
Math 591 - Real Analysis
Outline

1. Field Axioms
2. Elementary Propositions
3. Ordered Fields
4. Simple Propositions Concerning Ordered Fields
5. Further Algebraic Structure
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We let $F$ be a set and we suppose that there are \textit{binary} operations $+$ and $\cdot$ defined (well-defined) on $F$. We assume that:

- (1) $+$ and $\cdot$ are commutative.
- (2) $+$ and $\cdot$ are associative.
- (3) There is an element $1_F \in F$ such that $x \cdot 1_F = 1_F \cdot x = x$ for all $x \in F$.
- (4) There is an element $0_F \in F$ such that $0_F + x = x + 0_F = x$ for all $x \in F$. 
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Initial Setup and Axioms

We let $F$ be a set and we suppose that there are binary operations $+$ and $\cdot$ defined (well-defined) on $F$. We assume that:

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2. $+$ and $\cdot$ are associative.
3. There is an element $1_F \in F$ such that $x \cdot 1_F = 1_F \cdot x = x$ for all $x \in F$.
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Remark

The elements $1_F$ and $0_F$ are the multiplicative identity and the additive identity, respectively. We usually denote them by 0 and 1. It is easy to prove that these identities are unique.
(5) Given any $x \in F$ there is a $x' \in F$ such that $x \cdot x' = x' \cdot x = 1$. We denote $x'$ by $x^{-1}$ and call it the multiplicative inverse of $x$.

(6) Given any $x \in F$ there is a $x'' \in F$ such that $x'' + x = x + x'' = 0$. We denote $x''$ by $-x$ and call it the additive inverse of $x$.

(7) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$. 
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Field Axioms, Continued

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- (6) Given any \( x \in F \) there is a \( x'' \in F \) such that \( x'' + x = x + x'' = 0 \). We denote \( x'' \) by \( -x \) and call it the additive inverse of \( x \).

- (7) \( x \cdot (y + z) = x \cdot y + x \cdot z \) for all \( x, y, z \in F \).
A field is a set $F$ with binary operations $+$ and $\cdot$ that satisfy (1)–(7). In other words, a field is a commutative ring in which every non-zero element has a multiplicative inverse.
Proposition 1

For any $x \in F$, $x \cdot 0 = 0$.

Proof.

We have $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$ and so
$0 = -(x \cdot 0) + (x \cdot 0) = -(x \cdot 0) + (x \cdot 0 + x \cdot 0) = 0 + x \cdot 0 = x \cdot 0$. $\square$
Proposition 2

**Proposition**

Let $x \in F \setminus \{0\}$. Then $(x^{-1})^{-1} = x$.

**Proof.**

Let $x$ be a non-zero element of $F$. Since $x^{-1} \in F$, it has an inverse and so

$$1 = x^{-1} \cdot (x^{-1})^{-1}.$$ 

Hence we have

$$x \cdot 1 = x \cdot \left[ x^{-1} \cdot (x^{-1})^{-1} \right] = (x^{-1})^{-1}$$

which finishes the proof.
We state the following without proof:

- \( \forall x, y \in F, (-x)y = -(xy) = x(-y). \)
- For \( x, y \in F \) (non-zero) we have \( \frac{x}{y} =: x \cdot y^{-1} \neq 0 \) and \( \left( \frac{x}{y} \right)^{-1} = \frac{y}{x}. \)
- For \( a, b \in F \setminus \{0\} \) we have \((ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} \).
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- For $a, b \in F \setminus \{0\}$ we have $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$. 
Other Propositions

We state the following without proof:

- $\forall x, y \in F, (-x)y = -(xy) = x(-y)$.
- For $x, y \in F$ (non-zero) we have $\frac{x}{y} =: x \cdot y^{-1} \neq 0$ and $(\frac{x}{y})^{-1} = \frac{y}{x}$.
- For $a, b \in F \setminus \{0\}$ we have $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$. 
We assume that \((F, +, \cdot)\) is a field.

**Order Axiom**

There is a subset \(F^+\) of \(F\), called the set of positive elements of \(F\) that satisfies

- If \(x, y \in F^+\), \(x + y, x \cdot y \in F^+\).
- Given any \(a \in F\), exactly one of the following is true:
  1. \(a \in F^+\)
  2. \(a = 0\)
  3. \(-a \in F^+\)
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Remark/Definition

**Negative Elements of** $F$

The negative elements of $F$ are defined to be the subset $F \setminus (F^+ \cup \{0\})$ of $F$ and are denoted by $F^-$. 
Proposition 1

Proposition

If \( a \in F^+ \), then \(-a \in F^-\).

Proof.

If \( a \in F^+ \), then \( a \neq 0 \). Further, \( a \notin F^- \) and we cannot have \(-a \in F^+\). If we did have \(-a \in F^-\), then we would have \(-a + a = 0 \in F^+\), a contradiction.
Proposition 2

Proposition

\[ 1 \in F^+ \]

Proof.

Suppose that \( 1 \in F^- \). Then \(-1 \in F^+ \). Let \( x \in F^+ \). Then \((-1)x \in F^+ \) and it follows that \( x + (-1)x \in F^+ \). Hence \( 0 \in F^+ \), a contradiction.
We state the following without proof:

1. \( F^- \neq \emptyset \)

2. If \( a \neq 0 \) is in \( F \), then \( a \neq -a \).
We state the following without proof:

1. $F^- \neq \emptyset$

2. If $a \neq 0$ is in $F$, then $a \neq -a$. 
Definition

Define $\leq$ on $F$ by $x \leq y$ if and only if $y - x \in F^+ \cup \{0\}$. The symbols $<$, $>$, and $\geq$ are defined in the usual way.
Why we have a total order

That a field $F$ that satisfies the order axiom is totally ordered follows from the following theorems (stated without proof).

**Theorem**

$\leq$ on $F$ satisfies: (1) $x \leq x$; (2) $x \leq y$ and $y \leq x$ imply that $x = y$; (3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

**Theorem**

For any $a, b \in F$ exactly one of the following holds: (1) $a < b$; (2) $a = b$; (3) $a > b$. 
Manipulation of Inequalities

Theorem

Let $a, b, c, d \in F$. Then

- $a > b$ and $c \geq d$ imply $a + c > b + d$.
- $a > b > 0$ and $c \geq d > 0$ imply $ac > bd$.
- $a > b$, $c < 0$ imply $ac < bc$. 
Absolute Value

Definition

\[ |a| = \begin{cases} 
  a & \text{if } a \geq 0 \\
  -a & \text{if } a < 0 
\end{cases} \]
Properties of Absolute Value

Theorem

1. $|a| = \max (a, -a)$
2. $|a| = |-a|$
3. $|ab| = |a| |b|$
4. $|a + b| \leq |a| + |b|$ (The triangle inequality)
5. $|a - b| \geq ||a| - |b||$
6. If $r > 0$, $|a - b| < r$ if and only if $a - r < b < a + r$. 
Integer Multiples of Elements of $F$

**Definition**

Given a positive integer $k$, we define $ka := a + \cdots + a$.

From this definition follows:

**Lemma**

If $k \neq \ell$ are positive integers, then $ka \neq \ell a$.

Finally, we can show

**Theorem**

A totally ordered field $F$ is infinite.