Example of Calculating Curvature

In this example curvature is calculated without reparametrizing the curve by arc length. The formula that allows us to do this is given in Theorem 1 on page 98. This theorem says that, for a curve \( r(t) \), the curvature \( \kappa_r(t) \) is given by

\[
\kappa_r(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}
\]

To see how this works (it's easy), we calculate the curvature of the helix defined by \( r(t) = (2 \cos(t), 3 \sin(t), t^2) \). (This curve is a type of elliptical helix.) I won't record all the details of the calculation but these details are easy for the reader to supply. First, the derivatives are:

\[
r'(t) = (-2 \sin(t), 3 \cos(t), 2t), \quad r''(t) = (-2 \cos(t), -3 \sin(t), 2)
\]

Next, we find the cross product of these derivatives:

\[
r'(t) \times r''(t) = (6 \cos(t) + t \sin(t), -4t \cos(t) + 4 \sin(t), 6)
\]

Now we find the norm of the cross product just calculated. It is

\[
||r'(t) \times r''(t)|| = \sqrt{62 + 26t^2 - 10(-1 + t^2) \cos(2t) + 20t \sin(2t)}
\]

Finally, we calculate the cube of the norm of the first derivative:

\[
||r'(t)||^3 = (4t^2 + 9 \cos(t)^2 + 4 \sin(t)^2)^{3/2}
\]

Putting the pieces together gives

\[
\kappa_r(t) = \frac{\sqrt{62 + 26t^2 - 10(-1 + t^2) \cos(2t) + 20t \sin(2t)}}{(4t^2 + 9 \cos(t)^2 + 4 \sin(t)^2)^{3/2}}
\]
Example of Unit Tangent, Principal Normal and Curvature

Consider the curve \( r(t) = ti + t^2 j + t^3 k \).

- **Unit Tangent:** The unit tangent vector is \( T(t) = \frac{r'(t)}{\|r'(t)\|} \). Since \( r'(t) = \langle 1, 2t, 3t^2 \rangle \), we have
  \[
  T(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1 + 4t^2 + 9t^4}}.
  \]

- **Principal Normal:** (See page 94) We calculate \( N(t) = \frac{r''(t)}{\|r'(t)\|} \). Using the result just above, this is easy although the algebra is a bit tedious. The result is
  \[
  N(t) = \left\langle \frac{t(2 + 9t^2)}{\sqrt{1 + 4t^2 + 9t^4}}, \frac{1 - 9t^4}{\sqrt{1 + 4t^2 + 9t^4}}, \frac{3(t + 2t^3)}{\sqrt{1 + 4t^2 + 9t^4}} \right\rangle.
  \]
  Note that, in order to calculate \( T' \), you need to use the **quotient rule** in each component.

- **The Curvature:** We calculate the curvature using the formula from page 98: \( \kappa(t) = \frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3} \). Of course, \( r'(t) = \langle 1, 2t, 3t^2 \rangle \) and \( r''(t) = \langle 0, 2, 6t \rangle \). Then \( r'(t) \times r''(t) = \langle 6t^2, -6t, 2 \rangle \) and we obtain \( \kappa(t) = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \).