Limit Process Examples

We consider some simple examples that illustrate some of the basic processes seen in a calculus sequence. These are all limiting processes and can be visualized nicely using a program such as Maple. (Which is what was used to generate this document.)

1. One of the fundamental problems that the idea of a limit can solve is that of convergence, or, more specifically, the convergence of infinite sums of real numbers (or complex numbers, for that matter). For example, consider the infinite sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

The question that arises at once is whether or not we get a finite result when adding this infinite collection of positive real numbers. You can use a calculator or computer to try to get an idea of what happens. For instance, if we add the first 100 terms we get

$$\sum_{n=1}^{100} \frac{1}{n} \approx 5.187377518.$$ 

If we add the first 1000 terms we get

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.485470861.$$ 

Well, not much to be said yet... If we add the first 100,000 terms we get
\[
\sum_{n=1}^{100000} \frac{1}{n} = \Psi(100001) + \gamma \quad \text{at 10 digits} \rightarrow 12.09014612.
\]

The result is still bigger. What is happening? Do we end up seeing a trend in our sums that will tell us what the sum of all infinitely many of our numbers will be?

As another example of this type of process, let's look at

\[
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots
\]

The question is the same - do we see a trend in the sums as we add more and more of these numbers together? If we check the first 100 terms we have

\[
\sum_{n=1}^{100} \frac{1}{n} \quad \text{at 10 digits} \rightarrow 1.082322905.
\]

The sum of the first 1000 terms gives us

\[
\sum_{n=1}^{1000} \frac{1}{n} \quad \text{at 10 digits} \rightarrow 1.082323234.
\]

We already see a trend! We have the number 1.0823.

Let's try the first 5000 terms:

\[
\sum_{n=1}^{5000} \frac{1}{n} \quad \text{at 10 digits} \rightarrow 1.082323234.
\]

We now see the number 1.082323234. Two more:
at 20 digits \[ \sum_{n=1}^{1000000} \frac{1}{n} \approx 1.0823232337111378583 \]

at 20 digits \[ \sum_{n=1}^{1000000} \frac{1}{n} \approx 1.0823232337111381913 \]. Even better agreement between the last two. There is a definite trend in the sums and we'd say that the infinite sum of positive numbers above converges to 1.0823232337111, at least. (Note: \[ \sum_{n=1}^{\infty} \frac{1}{n} = \frac{\pi}{4} \].)

2. We can also look at area problems. Suppose that we want to find the area under the graph of \( y = 1 - x^2 \) between -1 and 1. The graph is

\[
> \text{plot}\left(1 - x^2, x=-1..1\right);
\]
This is not a "standard" region and so we cannot use standard formulas to find the area. However, we can approximate the area using rectangles. See the picture below.
Of course, we can include more rectangles:
It stands to reason that the sum of the areas of the rectangles in the second picture is a better approximation to the exact area than the sum of the rectangles in the first. We can, of course, continue this process and get arbitrarily better approximations. (Well, we hope so, any way.) This should give us a limiting value; i.e. a trend in the areas that lead us to the exact area. (Here it's $\frac{4}{3}$.)

Another, animated look at area is seen below. As the animation proceeds, we see that the rectangles do a better and better job at filling in the area under the curve. If this process were continued forever, we should get a limiting value.
with(Student[Calculus1]) :
ApproximateInt(\sin(x), 0 .. \pi, output = animation, partition = random, refinement = random, subpartition = width, iterations = 100, showpoints = false, boxoptions = [filled = [color = pink, transparency = 0.5]]);
An Approximation of the Integral of 
\( f(x) = \sin(x) \) 
on the Interval \([0, \pi]\)
Using a Midpoint Riemann Sum

Area: 2.054823246

3. We can also look at another geometric problem - approximating the tangent line to a function at a point. (Physically, this amounts to finding the \textit{instantaneous speed} of an object.) Suppose that we have the function
$f(x) = \frac{\sin(x)}{3 \cdot x}$ and suppose that we are interested in the tangent line at $x = \frac{\pi}{3}$. What can be done? Well, we can draw lines that are "almost" the tangent line and try to see a trend in our graphs (or, if we are calculating the slopes of these lines as we go, we'd look for a trend in the slopes.)

We see here that we don't get very close to the point at $\frac{\pi}{3}$, but there does seem to be a trend. We can get closer:
There does indeed seem to be a trend in the tangent lines.

4. Limiting processes come up in many other places as well. Below we see an example of a snowflake fractal, the Koch snowflake. The "Number of Iterations" box tells Maple how far to go in the process. Of course, once again, we ask what the limiting trend may be.
5. Another place that limiting processes come up is in the study of iterated maps. Consider the function $f(x) = r \cdot x \cdot (1 - x)$ where $r$ is a "parameter". What we study is the behavior of the sequence $f(a), f(f(a)), f(f(f(a))), f(f(f(f(a)))), \ldots$ where $a$ is some number in $[0, 1]$. A way to visualize this sequence is via "cobweb diagrams". The following code generates these diagrams:

```maple
mapiter := proc(F, x0, a, b, skipiter, plotiter, caption)
local xp, k, p, orbit, curve, diag;
xp := x0:
p := array(0..plotiter);
for k from 1 to skipiter do xp := F(xp); od;
for k from 0 to plotiter do p[k] := xp; xp := F(xp); od;
curve := plot(F(x), x = a..b, color = blue);
diag := plot(x, x = a..b, color = black);
orbit := plot(seq(op([[p[k-1], p[k]], [p[k], p[k]]]), k = 1..plotiter), color = red);
plots[display]([[orbit, curve, diag], title = caption, scaling
```
We will consider $f(x) = 3.78 \cdot x \cdot (1 - x)$.

$$f := x \mapsto 3.78 \cdot x \cdot (1 - x);$$

$$x \mapsto 3.78 \, x \, (1 - x)$$ \hfill (1)

We will generate a diagram for $a = 0.1$.

mapiter($f$, 0.1, 0, 1, 0, 100, "r = 3.78");
$r = 3.78$
Is there a trend??