Curve Analysis Examples

Example 1:

We consider the function \( f(x) = \frac{x^2}{x^3 - 8} \). We will do the following:

1. Find the domain of the function.
2. Find all horizontal/skew, and vertical asymptotes.
3. Find all critical points of the function.
4. Find the intervals on which the function is increasing or decreasing.
5. Find all intervals on which the function is concave up or concave down.
6. Identify all local extremes and inflection points.
7. Plot these points as well as any intercepts that we can find.
8. Plot a careful graph of the function, labelling all important points.

For (1), we note that the domain of the function is \((-\infty, 2) \cup (2, \infty)\).

For (2), since \( x^3 - 8 = (x - 2)(x^2 + 2x + 4) \), we see that there is a vertical asymptote at \( x = 2 \) (since the fraction cannot be reduced further.) Also, the horizontal asymptote is \( y = 0 \).

Now, to take care of (3), we need to find and simplify the derivative. This is done below using the computer:

```
In[10]:= f[x_] := x^2
     x^3 - 8

In[11]:= f'[x] // Simplify

    8 - x^3)^2
```

To find the critical points, we set \( f'(0) = 0 \) and solve. Here we see that \( x = 0 \) and \( x = \sqrt[3]{16} = 2 \sqrt[3]{2} = -2.51984 \). Next, we see that the derivative is undefined at \( x = 2 \), the vertical asymptote. Hence \( x = 2 \) is not a true critical point.

For item (4), we construct a table using the critical point(s) and the vertical asymptote(s).
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<table>
<thead>
<tr>
<th>Intervals</th>
<th>$(-\infty, -2.51984)$</th>
<th>$(-2.51984, 0)$</th>
<th>$(0, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Values</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$f'(x)$ (Test Values)</td>
<td>$-33/1225$</td>
<td>$5/27$</td>
<td>$-17/49$</td>
<td>$-129/361$</td>
</tr>
<tr>
<td>Increasing/Decreasing</td>
<td>Decreasing</td>
<td>Increasing</td>
<td>Decreasing</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>

In item (5), we need the second derivative, as we need to determine concavity. The second derivative is computed below:

$$f''[x] = \frac{2 \left(64 + 56x^3 + x^6\right)}{(-8 + x^3)^3}$$

We see that the second derivative is undefined at $x = 2$ and that the second derivative is zero when the numerator, $x^6 + 56x^3 + 64 = 0$. We can solve this by using the quadratic formula and we get $x = -3.7991, -1.05288$. In order to analyze the concavity of our function, we construct the same type of table as above, only using the second derivative on the test values. Note: **We must include the vertical asymptote in our table!! Concavity can change when going across a discontinuity!**

<table>
<thead>
<tr>
<th>Intervals</th>
<th>$(-\infty, -3.7991)$</th>
<th>$(-3.7991, -1.05288)$</th>
<th>$(-1.05288, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Values</td>
<td>-5</td>
<td>-2</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$f''(x)$ (Test Values)</td>
<td>$-17378/2352637$</td>
<td>$\frac{5}{32}$</td>
<td>$-\frac{1}{4}$</td>
<td>$\frac{5042}{177957}$</td>
</tr>
</tbody>
</table>

In item (6) we identify local extremes and inflection points. We see from the first derivative test that -2.51984 corresponds to a local minimum and that 0 corresponds to a local maximum.

**NOTE:** We used the first derivative test for the conclusions just above. We could have, just as easily, used the second derivative test. All we need to do is calculate the value of the second derivative at each critical point:

$$f''[-2.51984] = 0.0833336$$
Sure enough, we see that -2.51984 corresponds to a minimum (the test tells us that the curve is concave up there) and 0 corresponds to a maximum (the test tells us that the curve is concave down there).

As far as item (7) goes, both $x$ and $y$ intercepts are at the origin.

Finally we sketch a plot of the function and label all important points.

**Example 2:**

We will do the same as above with the function $g(x) = \ln(x^2 + 3x + 3)$.

What we'll do first is calculate and simplify both the first and second derivatives right away so that we can use them as needed. First we'll tell the computer what the function is and then find the derivatives.

\[
\ln(9) := g[x_] := \log(x^2 + 3x + 3)
\]

For the derivatives we have
\[ g'(x) \text{ Simplify} // \text{TraditionalForm} \]
\[
\frac{2x+3}{x^2+3x+3}
\]
\[ g''(x) \text{ Simplify} // \text{TraditionalForm} \]
\[
\frac{2x^2+6x+3}{(x^2+3x+3)^2}
\]

Now, note that the domain of our function is the entire real line. Also, there are now vertical or horizontal asymptotes.

**Critical Points**

To find critical numbers we solve \( f'(x) = 0 \) and determine where the derivative is undefined. It is clear that the derivative vanishes when \( x = -3/2 \) and, since there are no zeros in the denominator of the derivative (WHY?), this is the only critical point.

**Second Derivative Test on the Critical Points**

To apply the second derivative test, we calculate the value of the second derivative at each critical point. We have

\[ g''(-3/2) \]
\[ \frac{8}{3} \]

Since the value of the second derivative is positive, the point \((-3/2, -\ln(4/3))\) is a local minimum.

**Intervals where the function is increasing/decreasing**

We construct a table for this analysis:

<table>
<thead>
<tr>
<th>Intervals</th>
<th>((-\infty, -3/2))</th>
<th>((-3/2, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Values</td>
<td>-2</td>
<td>11</td>
</tr>
<tr>
<td>( g'(\text{Test Values}) )</td>
<td>-1</td>
<td>25/157</td>
</tr>
<tr>
<td>Increasing/Decreasing</td>
<td>Decreasing</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

**Concavity and Inflection Points**
To find possible inflection points, we set the second derivative equal to zero and determine if it’s undefined anywhere. For the roots of the second derivative, we have

\[
\text{In[77]}\ := \ \text{Solve}[g''[x] = 0, x]
\]

\[
\text{Out[77]}\ := \ \left\{ \left\{ x \to \frac{1}{2} \left(-3 - \sqrt{3}\right) \right\}, \left\{ x \to \frac{1}{2} \left(-3 + \sqrt{3}\right) \right\} \right\}
\]

\[
\text{In[78]}\ := \ % / \ N
\]

\[
\text{Out[78]}\ := \ \left\{ \left\{ x \to -2.36603 \right\}, \left\{ x \to -0.633975 \right\} \right\}
\]

Also, we see that the second derivative is never undefined (WHY?). To analyze concavity, we construct a table:

<table>
<thead>
<tr>
<th>Intervals</th>
<th>(-\infty, -2.36603)</th>
<th>(-2.36603, -0.633975)</th>
<th>(-0.633975, \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Values</td>
<td>-9</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>(g'') (Test Values)</td>
<td>-37/1083</td>
<td>1</td>
<td>-1/3</td>
</tr>
<tr>
<td>Concave Up or Concave Down</td>
<td>Concave Down</td>
<td>Concave Up</td>
<td>Concave Down</td>
</tr>
</tbody>
</table>

**The Plot**

\[
\text{In[82]}\ := \ \text{Plot}[g[x], \{x, -10, 10\}]
\]
Local Minimum at \(-3, -\ln(4/3)\)

Inflection Point at \((-2.36603, 0.40547)\)

Inflection Point at \((-0.633975, 0.405465)\)