The Derivative as a function

Recall the definition of the derivative at a point:

\[
\frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

If this limit exists, we say that \( f \) is differentiable at \( x = a \) and, of course, we remember that the derivative at a point is the slope of the tangent line at that point. For example, for the function

\[
f := x \to \exp\left( -\frac{x}{5} \right) \cdot \cos(x)
\]  
\[
= x \to e^{-\frac{1}{5}x} \cos(x),
\]

the slope of the tangent line at \( x = \frac{\pi}{3} \) is

\[
D(f) \left( \frac{\pi}{3} \right) = -0.7834839852.
\]

(You should do the numerical calculation on your calculator or by using a spreadsheet!) Now we can, in principle, evaluate this limit at every \( a \) that we need but that would be tedious, to say the least. What we want to do is develop the derivative as a function of the variable \( x \) so that we can use the derivative as a function, inputting \( x \) and outputting the slope of the curve at \( x \). How is this done? The first step is to look at a graph of the function and decide where the derivative is positive, negative, zero (or undefined). We will look at the graph of our function quite carefully, using a built-in Maple command.

\[
> \text{with(Student[Calculus1])}:
\]
\[
> \text{FunctionChart}(f(x), x = 0..\pi, \text{slope} = \{ \text{thickness}(3, 1), \text{linestyle}(\text{solid, dash}) \}, \text{concavity} = \text{arrow}, \text{pointoptions} = \{ \text{symbolsize} = 20 \});
\]
The Chart of  
\[ f(x) = \exp(-1/5x)\cos(x) \]
on the Interval \([0, 4\pi]\)
We'll ignore the arrows for now though they tell us other information about the function. The function is decreasing on the interval where the graph is the thin curve and increasing where the graph is the thick curve. Now take a look at the first segment of thin curve. The function is decreasing here and, if you draw in a tangent line, it has negative slope. **We see that when the function is decreasing, the derivative function must be negative.** Moreover, consider what is happening to the slope of the curve to the left and right of the first green cross. To the left of this cross we see that the slope is getting more and more negative as we move to the right towards the x-value of the green cross. Once we pass this x-value, however, the slope, still negative, becomes less so until we reach the minimum on the curve at the first green diamond. **We conclude: The derivative function is negative but changes from decreasing to increasing at the x-value of the first green cross and increases to zero at the first green circle (why zero??).**

We now move to the portion of the curve between the first green diamond and the second green diamond. The curve here is thick, denoting an interval where the function is increasing. Note that the derivative is always positive on this interval. **We see that when the function is increasing, the derivative function is positive.** Now concentrate on the part of the thick curve between the green diamond and the green cross. The slope is positive and getting steeper; i.e. the derivative is positive and increasing. Between the green cross and the second green diamond, the slope is still positive but it is getting shallower; i.e. the derivative is positive and decreasing. **Conclusion: The derivative function is positive and changes from increasing to decreasing at the x-value corresponding to the second green cross after which the slope decreases to zero at the second green diamond (why??).**

Between the second and third green diamonds, where the function is decreasing, we see that, between the diamond and the third green cross, the slope is negative and decreasing. From the cross to the diamond, the slope is still negative but is increasing (to zero). A similar analysis holds for the last segment of the curve.

Based on these observations, we see that a plot of the derivative should be:
The function and the derivative function plotted together on the same coordinate system show us the
correlation:

The blue curve here is the derivative curve (as should be clear after a moment's reflection). It is the graph of the derivative function $f'(x)$.

Let's look at another example. Take for this example the function $g(x) = x^3 \cdot e^{-\frac{1}{5} \cdot x^2}$. We will do the same type of analysis as above. First we'll tell Maple what the function is:

```maple
> g := x -> x^3 \cdot \exp\left(-\frac{1}{5} \cdot x^2\right);
```

Now that Maple knows our function, we can generate a "function chart" as above:

```maple
> FunctionChart(g(x), x = -9 .. 9, slope = [thickness(3, 1), linestyle(solid, dash)], concavity = arrow, pointoptions = [symbolsize = 20]);
```
We see that the function decreases from \(-\infty\) to the minimum between -4 and -2, increases from this minimum to the maximum between 2 and 4, and then decreases again from this maximum to \(+\infty\).

Hence we can say that the derivative is negative from \(-\infty\) to the minimum, positive between the minimum and maximum, and then negative again. Also, on the far left, the derivative, while negative, is also close to zero. The same happens on the far right. Hence, from left to right, the derivative function starts out negative but very close to the x-axis, becomes more negative (reaching the most negative point at about -4 - why?), turns around and reaches the x-axis at the x-coordinate of the minimum (why?), becomes positive (reaching it's most positive point at about -2 - why?) and then comes back down to zero at the origin (why?), increases again until reaching it's most positive point at about 2 (why?), turns
back and crosses the x-axis at the x-coordinate (why?) of the maximum, becomes negative and reaches it's most negative point at about 2 (why?), turns around and approaches the x-axis from below as x increases. We see that the following must be the plot of the derivative of $g(x)$:

We can directly compare $g(x)$ and the derivative curve to see that the plot just above is correct:
**Definition:** The derivative function $f'(x)$ is defined by $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ for any $x$ for which the limit exists.

**Example:** Which curve in the following plot is the derivative?
Example: Which curve in the following plot is the derivative?
The vertical red lines are vertical asymptotes.