Assignment 4 Solutions

Problem 1

We solve the initial value problem \( y'' + 2 y' + y = (x^2 + 1) e^{-x} \), \( y(0) = 1 \), \( y'(0) = 1 \).

Solution:

We first solve the homogeneous problem. Here, the characteristic equation is \( r^2 + 2 r + 1 = (r + 1)^2 \). Hence the eigenvalue is \( r = 1 \) with multiplicity 2 and the homogeneous solution is \( y_h = c_1 e^{-x} + c_2 x e^{-x} \).

To find the particular integral, we set up the initial version - \( y_p = (A x^2 + B x + C) e^{-x} \). However, we need to modify this expression as two of the terms (the last two) appear in the homogeneous solution. Multiplying by \( x^2 \) will remove these terms from the homogeneous solution and so out final version of \( y_p \) is

\[
y_p = (A x^4 + B x^3 + C x^2) e^{-x}.
\]

The next thing we do is take this function and plug it into the DE. I'll do this with the computer as follows. First, I'll define the function for Mathematica:

\[
y_p[x_] := (A x^4 + B x^3 + C x^2) e^{-x}
\]

Now we'll have the software compute \( y_p'' + 2 y_p' + y_p - (x^2 + 1) e^x \); this will make the determination of the equations for \( A \), \( B \), \( C \) easier for the software and the user.

\[
\text{plug} = y_p''[x] + 2 y_p'[x] + y_p[x] - (x^2 + 1) e^{-x} // \text{Simplify}
\]

\[
e^{-x} (-1 + 2 C + 6 B x - x^2 + 12 A x^2)
\]

With this in hand, we'll extract the coefficients and determine our equations.

First, we'll extract the coefficients to \( x^2 e^{-x} \):

\[
\text{Coefficient}[\text{plug}, x^2 e^{-x}]
\]

\[
-1 + 12 A
\]

Our first equation is \( 12 A - 1 = 0 \) (why?).

\[
\text{eqn1} := 12 A - 1 = 0
\]

Next, we'll extract the coefficient of \( x e^{-x} \):

\[
\text{Coefficient}[\text{plug}, x e^{-x}]
\]

\[
6 B
\]

Our second equation is \( 6 B = 0 \).

\[
\text{eqn2} := 6 B = 0
\]

Finally, the coefficients of \( e^{-x} \) give, by inspection, the equation \( 2 C - 1 = 0 \):

\[
\text{eqn3} := 2 C - 1 = 0
\]

At last, we can solve the three equations:
\[ \text{Solve}[[\text{eqn1}, \text{eqn2}, \text{eqn3}], \{A, B, C\}] \]

\[ \left\{ \left[ A \rightarrow \frac{1}{12}, B \rightarrow 0, C \rightarrow \frac{1}{2} \right] \right\} \]

It follows that \( y_p = \left( \frac{1}{12} x^4 + \frac{1}{2} x^2 \right) e^{-x} \) and the general solution to the DE is

\[ y = c_1 e^{-x} + c_2 x e^{-x} + \left( \frac{1}{12} x^4 + \frac{1}{2} x^2 \right) e^{-x}. \]

The last detail is to find the constants of integration. To do this with the computer, we'll input the solution as a function of \( x \) and then use it to get the two equations that will determine the integration constants.

Defining the function follows:

\[ y[x_] := c_1 e^{-x} + c_2 x e^{-x} + \left( \frac{1}{12} x^4 + \frac{1}{2} x^2 \right) e^{-x} \]

We'll have Mathematica solve our two equations:

\[ \text{Solve}[[y[0] = 1, y'[0] = 1], \{c_1, c_2\}] \]

\[ \{c_1 \rightarrow 1, c_2 \rightarrow 2\} \]

The solution to the initial value problem is: \( y = e^{-x} + 2 x e^{-x} + \left( \frac{1}{12} x^4 + \frac{1}{2} x^2 \right) e^{-x}. \)

**Problem 2**

The characteristic equation for the homogeneous problem is \( r^2 + 4 = 0 \) and so \( r = \pm 2i \) and the homogeneous solution is \( y_h = c_1 \cos(2x) + c_2 \sin(2x). \)

The first version of the particular solution is \( y_p = A \sin(2x) + B \cos(2x) + Cx^3 + Fx^2 + Gx + H. \) Since the sine and cosine terms appear in the homogeneous solution, we modify the first two terms to obtain

\[ y_p = A x \sin(2x) + B x \cos(2x) + C x^3 + F x^2 + G x + H \]

as the final version of the particular integral. We now plug this into the DE. To do this, we'll define the function for Mathematica:

\[ y_p[x_] := A \times \text{Sin}[2 \times] + B \times \text{Cos}[2 \times] + C \times^3 + F \times^2 + G \times + H \]

We'll now calculate as in the first problem, have the software compute \( y'' + 4 y - 3 \sin(2x) - x^3: \)

\[ \text{plugin2} = y_p''[x] + 4 y_p[x] - 3 \text{Sin}[2 \times] - x^3 // \text{Simplify} \]

\[ 2 F + 4 H + 6 C x + 4 G x + 4 F x^2 - x^3 + 4 C x^3 + 4 A \text{Cos}[2 \times] - (3 + 4 B) \text{Sin}[2 \times] \]

We'll now extract the coefficients and determine our equations.

1. We'll find the coefficient to cosine:

\[ \text{eqn1} = \text{Coefficient}[\text{plugin2}, \cos(2 \times)] = 0 \]

\[ 4 A = 0 \]

2. Now for the sine:
3. We now find the coefficients to the cubic term:

\[ \text{eqn3} = \text{Coefficient}[\text{plugin2}, x^3] = 0 \]
\[-1 + 4 C = 0 \]

4. Quadratic term:

\[ \text{eqn4} = \text{Coefficient}[\text{plugin2}, x^2] = 0 \]
\[4 F = 0 \]

5. Linear coefficients give:

\[ \text{eqn5} = \text{Coefficient}[\text{plugin2}, x] = 0 \]
\[6 C + 4 G = 0 \]

6. Finally the last equation comes from the constant terms:

\[ \text{eqn6} = 2 F + 4 H = 0 \]
\[2 F + 4 H = 0 \]

Finally, we solve the six equations for our coefficients:

\[
\text{Solve}[\{\text{eqn1, eqn2, eqn3, eqn4, eqn5, eqn6}\}, \{A, B, C, F, G, H\}]
\[
\{\{A \to 0, B \to -\frac{3}{4}, C \to \frac{1}{4}, F \to 0, G \to -\frac{3}{8}, H \to 0\}\}
\]

The general solution is:

\[ y = c_1 \sin(2x) + c_2 \cos(2x) - \frac{3}{4} x \cos(2x) + \frac{1}{4} x^3 - \frac{3}{8} x \]

**Problem 3**

The characteristic equation is \( r^2 + r - 4 = 0 \) and the eigenvalues are \( r = -\frac{1}{2} \pm \frac{\sqrt{17}}{2} \) and so the homogeneous solution is

\[ y_h = c_1 \exp\left(\left(-\frac{1}{2} + \frac{\sqrt{17}}{2}\right)x\right) + c_2 \exp\left(\left(-\frac{1}{2} - \frac{\sqrt{17}}{2}\right)x\right) \]

The particular solution takes the form

\[ y_p = (A x + B) e^{2x} \cos(x) + (C x + G) e^{2x} \sin(x) \]

No modification is needed. We'll define \( y_p \) for *Mathematica* and then carry out the calculations that are needed.

\[ y_p[x] := (A x + B) \exp[2 x] \cos[x] + (C x + G) \exp[2 x] \sin[x] \]

We'll plug this function into the DE as we did above:
\[ \text{plugin3} = y_p''[x] + y_p'[x] - 4 y_p[x] - x \exp[2 x] \cos[x] \] // Simplify

\[ e^x \left( (B + 2 C + 5 G - x + 5 C x + A (5 + x)) \cos[x] + (-5 B + 5 C + G + C x - A (2 + 5 x)) \sin[x] \right) \]

Now we'll extract the coefficients.

1. The equation determined by the coefficients of \( x e^{2x} \cos(x) \) is
   \[
   \text{eqn1} = \text{Coefficient[plugin3, } x \exp[2 x] \cos[x]) = 0
   \]
   \[-1 + A + 5 C = 0
   \]

2. We now get the equation determined by the coefficients of \( x e^{2x} \sin(x) \):
   \[
   \text{eqn2} = \text{Coefficient[plugin3, } x \exp[2 x] \sin[x]) = 0
   \]
   \[-5 A + C = 0
   \]

3. Now for the coefficients to \( e^{2x} \cos(x) \):
   \[
   \text{Coefficient[plugin3, Exp[2 x] Cos[x]]}
   \]
   \[5 A + B + 2 C + 5 G - x + A x + 5 C x
   \]
   Our third equation is therefore
   \[
   \text{eqn3} = 5 A + B + 2 C + 5 G = 0
   \]
   \[5 A + B + 2 C + 5 G = 0
   \]

4. We do the same for \( e^{2x} \sin(x) \).
   \[
   \text{Coefficient[plugin3, Exp[2 x] Sin[x]]}
   \]
   \[-2 A - 5 B + 5 C + G = 0
   \]
   \[eqn4 = -2 A - 5 B + 5 C + G = 0
   \]
   We now solve the equations for the coefficients:
   \[
   \text{Solve[([eqn1, eqn2, eqn3, eqn4], } \{A, B, C, G\})
   \]
   \[\{\{A \rightarrow \frac{1}{26}, B \rightarrow \frac{25}{169}, C \rightarrow \frac{5}{26}, G \rightarrow -\frac{49}{338}\}\}\]
   The general solution is therefore
   \[
   y = c_1 \exp\left(\frac{1}{2} + \frac{\sqrt{17}}{2} x\right) + c_2 \exp\left(-\frac{1}{2} - \frac{\sqrt{17}}{2} x\right) + \frac{1}{26} x + \frac{25}{169} e^{2x} \cos(x) + \left(\frac{5}{26} x - \frac{49}{338}\right) e^{2x} \sin(x)
   \]
   Finally, we solve the initial value problem. We'll define the general solution for Mathematica and then solve the two equations. The function is
   \[
   y[x_] :=
   \]
   \[c_1 \exp\left(\frac{1}{2} + \frac{\sqrt{17}}{2} x\right) + c_2 \exp\left(-\frac{1}{2} - \frac{\sqrt{17}}{2} x\right) + \frac{1}{26} x + \frac{25}{169} e^{2x} \cos[x] + \left(\frac{5}{26} x - \frac{49}{338}\right) e^{2x} \sin[x] \]
Now we solve the two equations:

\[
\text{Solve}[\{y[0] = 1, \ y'[0] = 1\}, \{c_1, \ c_2\}]
\]

\[
\left\{ \begin{array}{l}
\quad \quad \quad \quad c_1 \to \frac{1224 + 209 \sqrt{17}}{2873}, \quad c_2 \to -\frac{209 + 72 \sqrt{17}}{169 \sqrt{17}}
\end{array} \right\}
\]

The solution is:

\[
y = \left( \frac{1224 + 209 \sqrt{17}}{2873} \right) \exp \left( -\frac{1}{2} + \frac{\sqrt{17}}{2} \right) x + \left( -\frac{209 + 72 \sqrt{17}}{169 \sqrt{17}} \right) \exp \left( -\frac{1}{2} - \frac{\sqrt{17}}{2} \right) x + \left( \frac{1}{26} x + \frac{25}{169} \right) e^{2 x} \cos(x) + \left( \frac{5}{26} x - \frac{49}{338} \right) e^{2 x} \sin(x).
\]

**Problem 4**

The particular solution has the form \( y_p = v_1(x) x + v_2(x) x \ln(x) \) where

\[
v_1(x) = -\int \frac{1}{x} \ln(x) \ dx = -\int \ln(x) \ dx = -\frac{(\ln(x))^2}{2}
\]

and

\[
v_2(x) = \int \frac{x}{x} \ dx = \ln(x).
\]

The general solution is \( y = c_1 x + c_2 x \ln(x) + \frac{1}{2} (\ln(x))^2 \).

**Problem 5**

The homogeneous solution is \( y = c_1 e^x + c_2 e^{2x} \) and so we have \( y_p = v_1 e^x + v_2 e^{2x} \) where

\[
v_1 = -\int e^{2x} \cos(e^{-x}) \ dx = \int e^{-x} \cos(e^{-x}) \ dx = -\sin(e^{-x})
\]

and

\[
v_2 = \int e^{4x} \cos(e^{-x}) \ dx = \int e^{2x} \cos(e^{-x}) \ dx = -\cos(e^{-x}) - e^{-x} \sin(e^{-x}).
\]

The general solution is \( c_1 e^x + c_2 e^{2x} - e^x \sin(e^{-x}) + e^{2x}(-\cos(e^{-x}) - e^{-x} \sin(e^{-x})) \).