Differential Geometry - Midterm Exam

Due: 3/5 by 4pm.

A computer/Maple is not to be used for these problems!

(1) Let $\beta : \mathbb{R} \to \mathbb{R}^3$ be a smooth curve such that $\frac{d^3\beta}{dt^3} = 0$. Show that $\beta$ is contained in a plane.

(2) Let $E$ be the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$, where $a, b > 0$.
   (a) Find a parametrization of $E$.
   (b) Compute the curvature at $(a, 0)$ and $(0, b)$.

(3) Let $\gamma : I \to \mathbb{R}^3$ and $\tilde{\gamma} : I \to \mathbb{R}^3$ be unit-speed curves with nonzero curvature and Frenet-Serret frames \{T, N, B\} and \{\tilde{T}, \tilde{N}, \tilde{B}\}, respectively. Suppose that $\tilde{N}(s) = N(s)$ and $\tilde{B}(s) = B(s)$, for all $s \in I$, and $\tilde{\gamma}(0) = \gamma(0)$. Show that $\tilde{\gamma} = \gamma$.

(4) Let $\sigma : U \to \mathbb{R}^3$ be defined by
   $$\sigma(u,v) = (u \cosh v, u \sinh v, u^2),$$
   where $U = \{(u, v) \in \mathbb{R}^2 : u > 0\}$. Show that $\sigma$ is a regular coordinate patch for the hyperbolic paraboloid defined by $z = x^2 - y^2$.

(5) Let $M$ be the surface defined by $z = x^2 + 3xy - 5y^2$.
   (a) Show that at $p = (0, 0, 0)$ the unit vectors $u = (1, 0, 0)$ and $v = (0, 1, 0)$ are a basis for $T_p M$.
   (b) Show that $\alpha_1(t) = (t, 0, t^2)$ and $\alpha_2(t) = (0, t, -5t^2)$ define paths on $M$, passing through $p$, and with tangents $u$ and $v$ respectively at $p$.
   (c) Find a unit normal field defined on a neighborhood of $p$ in $M$.
   (d) Compute the derivatives $\nabla_u U$ and $\nabla_v U$, where $U$ is the unit normal field computed in part (c) and $u$ and $v$ are the tangent vectors in (a).
   (e) Show that with respect to the basis given in (a) the shape operator at $p$ is given by the matrix $S_p = \begin{pmatrix} 2 & 3 \\ 3 & -10 \end{pmatrix}$.
   (f) Compute the normal curvatures in the directions $u$ and $v$ where $u$ and $v$ are the tangent vectors in (a).