**Eigenvalues and Eigenvectors: Examples**

We consider here several examples of computing eigenvalues and eigenvectors for matrices. Recall that, in order to find the eigenvalues of a matrix $A$, we have to find the zeros of its characteristic polynomial $f(\lambda) = \det(A - \lambda I)$. Once the eigenvalues have been found, we then find the eigenvectors.

**Example 1:** We consider the matrix

$$A = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix}.$$

### Eigenvalues

The characteristic polynomial is

$$\det \begin{pmatrix} 7 - \lambda & -1 & 6 \\ -10 & 4 - \lambda & -12 \\ -2 & 1 & -\lambda - 1 \end{pmatrix}$$

We can expand this determinant about any row we wish. The expansion about the first row is:

$$(7 - \lambda) \det \begin{pmatrix} 4 - \lambda & -12 \\ 1 & -\lambda - 1 \end{pmatrix} - (-1) \det \begin{pmatrix} -10 & -12 \\ -2 & -\lambda - 1 \end{pmatrix} + 6 \det \begin{pmatrix} -10 & 4 - \lambda \\ -2 & 1 \end{pmatrix}$$

The end result is

$$\text{Expand}[ (7 - \lambda) (- (\lambda + 1) (4 - \lambda) + 12) + (10 (\lambda + 1) - 24) + 6 (-10 + 2 (4 - \lambda)) ]$$

$$30 - 31 \lambda + 10 \lambda^2 - \lambda^3$$

Here's the characteristic polynomial:

$$f(\lambda) = -\lambda^3 + 10 \lambda^2 - 31 \lambda + 30.$$  

We need to find the zeros of this polynomial. In this case the problem is easy, as we can clearly write

$$f(\lambda) = -(\lambda - 2)(\lambda - 3)(\lambda - 5)$$

Our eigenvalues are therefore $\lambda = 2, 3, 5$; i.e. there are three distinct eigenvalues. This is the easiest case.

### Eigenvectors

We find the eigenvectors by solving the system $A \mathbf{v} = \lambda \mathbf{v}$ for each eigenvalue.

1. We solve the system $A \mathbf{v} = 2 \mathbf{v}$ where we'll take $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The system of equations that we get is
We can solve this system to find that \( y = -x \) and \( z = -x \). Therefore our eigenvector has the form \( v = \begin{pmatrix} x \\ -x \\ -x \end{pmatrix} \), \( x \in \mathbb{R} \). We'll take \( x = 1 \) so that \( v = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \) is an eigenvector for \( \lambda = 2 \). Note that, in this case, all other eigenvectors for this eigenvalue are multiples of \( \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \).

2. We solve the system \( A \, v = 3 \, v \). We'll take \( v \) to have the same form as above. The resulting system of equations is

\[
\begin{align*}
4 \, x - y + 6 \, z &= 0 \\
-10 \, x + y - 12 \, z &= 0 \\
-2 \, x + y - 4 \, z &= 0
\end{align*}
\]

We solve this system to find \( y = -2 \, x \) and \( z = -x \). Hence, our eigenvectors here are \( \begin{pmatrix} x \\ -2 \, x \\ -x \end{pmatrix} \), \( x \in \mathbb{R} \). Taking \( x = 1 \) gives \( v = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \) and, again, all other eigenvectors for this eigenvalue are multiples of this vector.

3. For the system \( A \, v = 5 \, v \) we proceed in the same fashion to obtain the system

\[
\begin{align*}
2 \, x - y + 6 \, z &= 0 \\
-10 \, x + y - 12 \, z &= 0 \\
-2 \, x + y - 6 \, z &= 0
\end{align*}
\]

and the solutions to this system are \( y = -2 \, x \) and \( z = -\frac{2}{3} \, x \). Taking \( x = 3 \) we have \( v = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} \) (and all other eigenvectors for this eigenvalue are multiples of this vector).

We've now found all eigenvectors for this matrix.

Finally, let \( Q \) be the matrix \( Q = \begin{pmatrix} 1 & 1 & 3 \\ -1 & -2 & -6 \\ 1 & 1 & -2 \end{pmatrix} \) - the matrix whose columns are the eigenvectors for \( A \). We then have

\[
Q^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ -4 & -1 & -3 \\ 1 & 0 & 1 \end{pmatrix}
\]

Computing \( Q^{-1} A \) we have \( Q^{-1} A Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \); we've diagonalized \( A \). Though it ought to be clear, we note that the three eigenvectors we found are a basis for \( \mathbb{R}^3 \).
Example 2: Find the eigenvalues and corresponding eigenvectors for $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix}$.

**Eigenvalues**

The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} 3 - \lambda & 1 & -1 \\ 1 & 3 - \lambda & -1 \\ 3 & 3 & -1 - \lambda \end{pmatrix} = (3 - \lambda)(-3 - \lambda)(\lambda + 1) + 3 - (1 + \lambda) + 3 - (3 - 2 - \lambda))$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Hence, since $f(\lambda) = -(\lambda - 1)(\lambda - 2)^2$, the eigenvalues are $\lambda = 2, 2, 1$. Here, then, we have a repeated eigenvalue.

**Eigenvectors**

1. For the repeated eigenvalue $\lambda = 2$, we have the system of equations

$$x + y - z = 0$$
$$x + y - z = 0$$
$$3x + 3y - 3z = 0$$

where we've taken the eigenvector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, as before. By inspection, our system consists of only one equation: $x + y - z = 0$.

Note, though, that we have two distinct solutions for this system: $x = 1, y = -1, z = 0$ and $x = 1, y = 0, z = 1$ and these two solutions give us two linearly independent eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

2. For $\lambda = 1$, we have an eigenvector $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as you can easily check.

Finally, let $Q = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Then $Q^{-1} = \begin{pmatrix} -1 & -2 & 1 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$ and $Q^{-1}AQ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Therefore $A$ is diagonalizable. Again, the eigenvectors we found form a basis for $\mathbb{R}^3$.

Example 3: We consider the matrix $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$.

**Eigenvalues**
The characteristic polynomial is \( f(\lambda) = (\lambda - 2)(\lambda - 4) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 \). The eigenvalues are \( \lambda = 3, 3 \); a repeated eigenvalue.

### Eigenvectors

Letting \( v = \begin{pmatrix} x \\ y \end{pmatrix} \), we have the system \( \begin{cases} x + y = 0 \\ -x - y = 0 \end{cases} \) or the equation \( x + y = 0 \). This tells us that the only eigenvector is \( v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) (of course, all multiples of this vector are eigenvectors).

Since we are not able to find a second eigenvector linearly independent to \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), we cannot diagonalize this matrix.

This example shows that a repeated eigenvalue may **not** have a number of eigenvectors that matches its multiplicity. This often happens.