Gram-Schmidt Orthornormalization

Given a linearly independent subset $S = \{w_1, \ldots, w_n\}$ in an inner product space $V$ (not necessarily finite dimensional), we can construct an orthonormal set $S' = \{v_1, \ldots, v_n\}$ such that $\text{span}(S) = \text{span}(S')$ as follows:

- **Step 1:**
  
  We let $v_1 = \frac{w_1}{\|w_1\|}$.

- **Step 2:**
  
  Supposing that the vectors $v_1, \ldots, v_{k-1}$ have been constructed, we let
  
  $$v_k = \frac{w_k - \sum_{j=1}^{k-1} \langle w_k, v_j \rangle v_j}{\|w_k - \sum_{j=1}^{k-1} \langle w_k, v_j \rangle v_j\|}$$

  We then obtain, for $k = 1, \ldots, n$, a collection of mutually orthogonal unit vectors; i.e. an orthonormal set.

- **Examples:**

  1. Consider the real vector space $C([0, 2])$ with the inner product $\langle f, g \rangle = \int_0^2 f(t) g(t) \, dt$. Let $S = \{1, x, x^2 + x, x^3 + 3\}$. We construct an orthonormal set from $S$. First, for $w_1 = 1$, we calculate

  $$v_1 = \frac{1}{\sqrt{\int_0^2 1^2 \, dx}}$$

  $$\frac{1}{\sqrt{2}}$$

  Now, for the vector $w_2 = x$, we have

  $$v_2 = \frac{x - v_1 \int_0^2 t \, v_1 \, dt}{\sqrt{\int_0^2 (x - v_1 \int_0^2 t \, v_1 \, dt)^2 \, dx}}$$

  $$\frac{3}{2} (-1 + x)$$

  To find $v_3$, we calculate
\[ v_3 = \frac{(x^2 + x) - v_2 \int_0^1 (x^2 + x) \, dx - v_1 \int_0^1 (x^2 + x) \, dx}{\sqrt{\int_0^1 ((x^2 + x) - v_2 \int_0^1 (x^2 + x) \, dx - v_1 \int_0^1 (x^2 + x) \, dx)^2 \, dx}} \]  
\[ \text{FullSimplify} \]
\[ \frac{1}{2} \sqrt{\frac{5}{2}} (2 + 3 (-2 + x) x) \]

Finally, for \( v_4 \), we have

\[ v_4 = \frac{x^3 + 3 - v_1 \int_0^1 (x^3 + 3) \, dx - v_2 \int_0^1 (x^3 + 3) \, dx - v_3 \int_0^1 (x^3 + 3) \, dx}{\sqrt{\int_0^1 ((x^3 + 3) - v_1 \int_0^1 (x^3 + 3) \, dx - v_2 \int_0^1 (x^3 + 3) \, dx - v_3 \int_0^1 (x^3 + 3) \, dx)^2 \, dx}} \]  
\[ \text{FullSimplify} \]
\[ \frac{1}{2} \sqrt{\frac{7}{4377}} (-1 + x) (2 + 5 (-2 + x) x) \]

We can check the inner products to see that we do have an orthonormal set.

\[ \int_0^2 v_1 v_2 \, dx \]
\[ 0 \]

\[ \int_0^2 v_1 v_3 \, dx \]
\[ 0 \]

\[ \int_0^2 v_1 v_4 \, dx \]
\[ 0 \]

\[ \int_0^2 v_2 v_3 \, dx \]
\[ 0 \]

\[ \int_0^2 v_2 v_4 \, dx \]
\[ 0 \]

\[ \int_0^2 v_3 v_4 \, dx \]
\[ 0 \]

Sure enough, we have constructed the O.N. set

\( \left\{ \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \sqrt{\frac{7}{4377}} \right\} \)