**Linear Transformations**

**DEFINITION:** A linear transformation (or linear map) from the vector space $V$ to the vector space $W$ (both over the same field) is a function $T: V \rightarrow W$ such that

- For all $x, y \in V$, $T(x + y) = T(x) + T(y)$
- For all $a \in F$ and all $x \in V$, $T(ax) = aT(x)$.

**NOTE:** Take note of the properties of a linear map listed on page 65.

**Examples:**

- Define $T_\theta(x, y) = (x \cos(\theta) - y \sin(\theta), x \sin(\theta) + y \cos(\theta))$ for any $(x, y) \in \mathbb{R}^2$. Then, for $\theta \in \mathbb{R}$, $T_\theta$ is a linear map from the plane to itself that rotates a vector by the angle $\theta$.
- Define $T: \mathbb{F}^n \rightarrow \mathbb{F}^2$ by $T(a_1, a_2, \ldots, a_n) = (a_1, a_2, 0, \ldots, 0)$. Then $T$ is a linear map (the canonical projection) from $\mathbb{F}^n$ onto $\mathbb{F}^2$. There are other projections. What are they?
- Let $L^1((0, \infty))$ be the vector space of functions $f : (0, \infty) \rightarrow \mathbb{R}$ for which $\int_0^{\infty} |f(x)| \, dx < \infty$. Define $T : L^1((0, \infty)) \rightarrow \mathbb{R}$ by $T(f) = \int_0^{\infty} f(x) \, dx$. Then $T$ is a linear transformation or map.
- Let $C^1((a, b))$ be the vector space of functions that are once continuously differentiable on the open interval $(a, b)$. Let $C((a, b))$ be the vector space of functions that are continuous on the open interval $(a, b)$. Define $D : C^1((a, b)) \rightarrow C((a, b))$ by $D(f) = \frac{d}{dx} f$. Then $D$ is a linear transformation.
### Special Transformations

The **identity transformation** from a vector space $V$ to itself is the map $I_V: V \rightarrow V$ given by $I_V(x) = x$ for all $x \in V$.

The **zero transformation** is the transformation $T_0: V \rightarrow V$ defined by $T_0(x) = 0$ for all $x \in V$. 
Kernels and Ranges

**DEFINITION:** Let \( T : V \rightarrow W \) be a linear transformation. The **kernel** of \( T \) is defined by \( N(T) = \{ x \in V : T(x) = 0 \} \). The kernel is also called the **null space**. Note that we always have \( 0 \in N(T) \).

The **range** of \( T \) is defined by \( R(T) = \{ T(x) : x \in V \} \subseteq W \).

**Examples:**

1. Consider vector space \( V \) of convergent sequences (over \( \mathbb{R} \)). Define \( T : V \rightarrow V \) by \( T([a_n]) = (0, 0, a_2, a_3, \ldots) \). What is the kernel of \( T \)? The answer is obvious?? The range is also easy.

2. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) by \( T(x, y, z) = (x - y, y - z, x - z) \). What is \( N(T) \)?

Here we see that if \((a, b, c) \in N(T)\), then we must have \( a = b = c \). Hence \( N(T) = \{(a, a, a) : a \in \mathbb{R} \} \). The range of this transformation is \( \mathbb{R}^2 \).

3. Let \( A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be defined by \( A(x, y, z) = (x - y - z, 2x - 2y - 2z, 3x - 3y - 3z) \).

We have \( N(A) = \{a + b, b, a : a, b \in \mathbb{R} \} \).
Theorem 2.1

Let $V$ and $W$ be vector spaces and let $T : V \rightarrow W$ be a linear transformation. Then $N(T)$ and $R(T)$ are subspaces of $V$ and $W$, respectively.
Theorem 2.2

Let $V$ and $W$ be vector spaces and suppose that $T: V \rightarrow W$ be linear. If $\{v_1, \ldots, v_n\}$ is a basis for $V$, then

$$R(T) = \text{span}(T(v_1), \ldots, T(v_n)).$$
Nullity and Rank of a Linear Transformation

**DEFINITION:** Given a linear transformation $T$ from the vector space $V$ to the vector space $W$, the nullity of $T$ is the dimension of the kernel of $T$ (if finite dimensional) and the rank of $T$ is the dimension of the range of $T$ (if finite dimensional). We denote these dimensions by $\text{nullity}(T)$ and $\text{rank}(T)$, respectively.
Dimension Theorem (Theorem 2.3)

Let $V$ and $W$ be vector spaces. Let $T : V \rightarrow W$ be a linear map. If $V$ is finite dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$
Theorem 2.4

Let $V$ and $W$ be vector spaces and let $T : V \rightarrow W$ be linear. Then $T$ is one-to-one (or injective) if and only if $N(T) = \{0\}$. 
Theorem 2.5

Let $V$ and $W$ be vector spaces of equal finite dimension. Let $T : V \rightarrow W$ be linear. EOTFILETTO:

1. $T$ is one-to-one.
2. $T$ is onto (or surjective).
3. $\text{rank}(T) = \dim(V)$. 
Theorem 2.6

Let $V$ and $W$ be vector spaces over $F$. Suppose that \{v_1, \ldots, v_n\} be a basis for $V$. For $w_1, \ldots, w_n \in W$, there is a unique linear transformation $T : V \rightarrow W$ such that $T(v_j) = w_j$. 